

Monochromatic loose path partitions in k -uniform hypergraphs

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Abstract

A conjecture of Gyárfás and Sárközy says that in every 2-coloring of the edges of the complete k -uniform hypergraph K_n^k , there are two disjoint monochromatic loose paths of distinct colors such that they cover all but at most $k - 2$ vertices. A weaker form of this conjecture with $2k - 5$ uncovered vertices instead of $k - 2$ is proved, thus the conjecture holds for $k = 3$. The main result of this paper states that the conjecture is true for all $k \geq 3$.

Keywords: Colored complete uniform hypergraphs, monochromatic loose path, partition

1 Introduction

A hypergraph $H = (V, E)$ consists of a set V of vertices and a set E of edges, where each edge is a subset of V . If all the edges of H have same size k , then the hypergraph H is said to be k -uniform. Let K_n^k denote the complete k -uniform hypergraph on n vertices (the family of all k -element subsets of a n -element set). A k -uniform loose (or linear) path of length ℓ , denoted \mathcal{P}_ℓ^k , is a k -uniform hypergraph with edges e_1, e_2, \dots, e_ℓ such that $\forall i \in [\ell - 1], |e_i \cap e_{i+1}| = 1$ and $|e_i \cap e_j| = 0$ for all other pairs $\{i, j\}$, $i \neq j$. For a loose path \mathcal{P}_ℓ^k and a vertex $v \in V(\mathcal{P}_\ell^k)$, if v lies in two edges of \mathcal{P}_ℓ^k , then we call v a 2-degree vertex of \mathcal{P}_ℓ^k . A k -uniform tight path of length ℓ , is a sequence of $k + \ell$ vertices with every consecutive set of k vertices forms an edge. For $k = 2$ we obtain the usual definition of a path P_ℓ with ℓ edges.

In this paper r -coloring always means edge-coloring with r colors (traditionally red and blue when $r = 2$). The following simple proposition, introduced by Gerencsér and Gyárfás in [1], is our starting point here.

Proposition 1.1 *In any 2-coloring of the edges of a finite complete graph the vertices can be partitioned into a red and a blue path. Here the empty graph and the one-vertex graph is accepted as a path of any color.*

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Note that any result about covering the vertices of edge-colored graphs by a small number of monochromatic subgraphs will imply a Ramsey-type result as a corollary. For example, Proposition 1.1 implies the bound $R(P_n, P_m) \leq n + m - 3$ for $n, m \geq 2$. In fact, Proposition 1.1 subsequently gave birth to the area of partitioning edge-colored complete graphs into monochromatic subgraphs. There have been many further results, questions and conjectures in this area, many of which generalize Proposition 1.1 in graphs or hypergraphs. we refer to two surveys [2, 5]. However, in contrast to the graph case, there are only a few results on covering the vertices with monochromatic pieces of hypergraphs, see for example, [3, 4, 6, 7].

There are various definitions of paths and cycles (for example, Berge, loose and tight) for hypergraphs. We focus on loose path here. Similar to the graph case, a set of less than k vertices in an edge-colored k -uniform hypergraph is accepted as a loose path of any color. However, it seems difficult to extend Proposition 1.1 to loose or tight paths of hypergraphs. The following conjecture first presented by Gyárfás and Sárközy can be found in [2] and [3].

Conjecture 1.2 *In every 2-coloring of the edges of K_n^k there are two disjoint monochromatic loose paths of distinct colors covering all but at most $k - 2$ vertices. This estimate is sharp for sufficiently large n .*

Gyárfás and Sárközy [3] presented the following construction to show that if Conjecture 1.2 holds, then it is best possible for n large enough: Consider the complete k -uniform hypergraph K_n^k with vertex bipartitions Q and S , where $|Q| = (k - 1)m + 1$, $|S| = 2(k - 1)$ and $m \geq 4(k - 1)$. Then color all k -element subsets of Q red and all uncolored k -element subsets of $Q \cup S$ blue.

2 Partitions by monochromatic loose paths

In this section, we will prove the following slightly stronger result than Conjecture 1.2.

Theorem 2.1 *Suppose that the edges of the complete k -uniform hypergraph K_n^k are colored with two colors, where $n \equiv 2 \pmod{k - 1}$. Then $V(K_n^k)$ can be partitioned into two monochromatic loose paths of distinct colors.*

It is obvious that Theorem 2.1 implies Conjecture 1.2: For each $n \not\equiv 2 \pmod{k - 1}$, removing at most $k - 2$ vertices from K_n^k will obtain a smaller $K_{n'}^k$ with $n' \equiv 2 \pmod{k - 1}$, then by Theorem 2.1, $V(K_{n'}^k)$ can be partitioned into two monochromatic loose paths of distinct colors. That is, there are two disjoint monochromatic loose paths of distinct colors such that they cover all but at most $k - 2$ vertices of K_n^k .

Proof of Theorem 2.1. Suppose the assertion is false. Then take vertex disjoint red and blue loose paths P_R and P_B such that they cover as many vertices as possible, and subject to this, the difference between $|V(P_R)|$ and $|V(P_B)|$ is maximal. Let W be the set of vertices uncovered by the paths P_R and P_B . Without loss of generality suppose that $|V(P_R)| \geq |V(P_B)|$. Then we have the following claim:

Claim $|V(P_B)| = r(k - 1) + 1$ for some integer $r \geq 1$, that is, P_B is proper.

If P_B is not proper, then $|V(P_B)| \leq k - 1$. Note that now the red path P_R is proper and W is not empty. Then $|V(P_B)| + |W| = (k - 1)s + 1$ for some integer $s \geq 1$. Since

$|V(P_R)| + |V(P_B)|$ is maximal then $|V(P_B)| = k - 1$. Let $e = \{v_1, \dots, v_k\}$ be the last edge with an 2-degree vertex v_1 of P_R . Let $\{u_1, \dots, u_{k-1}\}$ be the vertex set of P_B . Then we have $|W| = 1$. Otherwise, $|W| = (s - 1)(k - 1) + 1$ for some integer $s \geq 2$. Let w_1, \dots, w_k be k vertices of W . Then both edges $\{v_k, u_1, \dots, u_{k-1}\}$ and $\{v_k, w_1, \dots, w_k\}$ are blue, hence the two edges form a new blue path, say P'_B . Then $P_R - e$ and P'_B cover more vertices, a contradiction. Let w be the unique vertex of W . Now we consider two cases as follows.

Case 1 $|V(P_R)| = k$, that is, P_R is induced by an edge.

Then $\{v_1, \dots, v_k\}$ is the unique edge of P_R . It is easy to check that for each $i \in [k]$, $\{v_i, u_1, \dots, u_{k-1}\}$ is blue. Then $\{v_2, \dots, v_k, w\}$ is red. Otherwise the two edges form a new blue path covering all vertices, a contradiction. Now a blue edge $\{v_1, u_1, \dots, u_{k-1}\}$ and a red edge $\{v_2, \dots, v_k, w\}$ cover all vertices of K_n^k , this contradicts the hypothesis.

Case 2 $|V(P_R)| = t(k - 1) + 1$ for some integer $t \geq 2$, that is, P_R contains at least two edges.

Let $f = \{x_1, \dots, x_k\}$ be the first edge with a 2-degree vertex x_k of P_R . Note that $f_1 = \{u_1, \dots, u_{k-1}, w\}$ is red and $f_2 = \{v_k, u_1, \dots, u_{k-1}\}$ is blue. Then $f_3 = \{x_1, v_3, \dots, v_k, w\}$ is blue too. Otherwise, the red path $P_R - e + f_3 + f_1$ together with the blue path $\{v_2\}$ cover all vertices of K_n^k . By symmetry $f_4 = \{v_2, x_2, \dots, x_{k-1}, w\}$ is blue. Now three edges f_2, f_3 and f_4 induce a blue path. The blue path together with the red path $P_R - e - f$ can cover all vertices of K_n^k . A contradiction. This completes the proof of the claim.

The claim means that both two paths are proper. Then $|W| = s(k - 1)$ for some $s \geq 1$, since $n \equiv 2 \pmod{k - 1}$. Let w_1, \dots, w_{k-1} be $k - 1$ vertices of W . We first show that P_R contains at least two edges. Otherwise, P_R induced by an edge $\{v_1, \dots, v_k\}$. Let $\{u_1, \dots, u_k\}$ be the unique edge of P_B . Then similar to above, both edges $\{v_k, u_2, \dots, u_k\}$ and $\{v_k, w_1, \dots, w_{k-1}\}$ are blue and hence form a blue path. The blue path together with the red path $\{v_1, \dots, v_{k-1}\}$ will cover more vertices, a contradiction.

Let $f = \{x_1, \dots, x_{k-1}, x_k\}$ and $e = \{v_1, v_2, \dots, v_k\}$ be the first and last edges of P_R respectively, where x_k and v_1 are two 2-degree vertices of P_R ($x_k = v_1$ is allowed). Let $g = \{u_1, u_2, \dots, u_k\}$ be the last edge of P_B . If P_B is of length at least two, then u_1 is a 2-degree vertex of P_B .

For convenience, let $X = \{x_1, \dots, x_{k-1}\}$, $V = \{v_2, \dots, v_k\}$, $U = \{u_2, \dots, u_k\}$ and $W' = \{w_1, \dots, w_{k-1}\}$. For each element $Y \in \{X, V, U, W'\}$, let Y_i denote an i -element subset of Y . Specially, let $Y_0 = \emptyset$. By the assumption of P_R and P_B , the following results are easy to check:

- (i) for $i \in [k] \setminus \{1\}$, $\{v_i\} \cup W'$ is blue, $\{u_i\} \cup W'$ is red;
- (ii) for $i, j \in [k] \setminus \{1\}$, $\{v_i, u_j\} \cup W'_{k-2}$ is blue; by symmetry, for $i \in [k - 1], j \in [k] \setminus \{1\}$, $\{x_i, u_j\} \cup W'_{k-2}$ is also blue;
- (iii) for $i \in [k - 1]$, $\{w_i\} \cup V$ is red; (otherwise, the blue path $P_B + \{u_k, v_k\} \cup W'_{k-2} + \{w_i\} \cup V$ together with the red path $P_R - e$ will cover more vertices, where $w_i \notin W'_{k-2}$);
- (iv) for $i \in [k - 1], j \in [k] \setminus \{1\}$, $\{x_i, v_j\} \cup W'_{k-2}$ is blue (otherwise, $P_R - e + \{x_i, v_j\} \cup W'_{k-2} + \{w_l\} \cup V$ is a longer red path, where $w_l \notin W'_{k-2}$);
- (v) for $i \in [k] \setminus \{1\}, j \in [k - 1]$, $\{v_i, w_j\} \cup X_{k-2}$ is blue (otherwise, the red path $P'_R = P_R - f + \{v_i, w_j\} \cup X_{k-2} + \{u_k\} \cup W'$ and the blue path $P'_B = P_B - g$ is a new covering with $|V(P'_R)| + |V(P'_B)| = |V(P_R)| + |V(P_B)|$ and $|V(P'_R)| - |V(P'_B)| > |V(P_R)| - |V(P_B)|$, a contradiction); by symmetry, for $i, j \in [k - 1]$, $\{x_i, w_j\} \cup V_{k-2}$ is also blue;

(vi) for $i, j \in [k] \setminus \{1\}$, $l \in [k-1]$, $\{v_i, u_j, w_l\} \cup X_{k-3}$ is blue (otherwise, the red path $P_R - e + \{v_i, u_j, w_l\} \cup X_{k-3} + \{x_s\} \cup U$ and the blue path $P_B - g$ is a new covering, where $x_s \notin X_{k-3}$. Similar to (v), we get a contradiction).

Then $P'_R = P_R - f - e$ and $P'_B = P_B + \{u_2, v_2, w_1\} \cup X_{k-3} + \{w_1, x_i\} \cup V_{k-2} + \{v_j, x_l\} \cup W'_{k-2}$ can cover more vertices than P_R and P_B , where $x_i, x_l \notin X_{k-3}$, $w_1 \notin W'_{k-2}$ and $v_j \in V_{k-2}$, a contradiction. This completes the proof.

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